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A simple example of modeling hybrid systems using bialgebras: preliminary version

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Hybrid systems

Let Σ be a finite alphabet, and let $W = \Sigma^*$ be the set of strings of letters of Σ. W is a semigroup with identity. A finite automaton over Σ is a finite set S of states together with a transition map $\delta : \Sigma \times S \to S$ and an initial state $s_0 \in S$. The transition map δ can be extended to a map $\delta : W \times S \to S$. If $s \in S$ and $w \in W$ we denote $\delta(w, s)$ by $w \cdot s$.

Let k be an algebraically closed field of characteristic 0. Suppose that for each $s \in S$ we have a pointed irreducible cocommutative k-bialgebra H_s with counit $\epsilon_s : H_s \to k$ and unit $\eta_s : k \to H_s$, an augmented commutative right H_s-module algebra R_s, and an observation $f_s \in R_s$. In this case $H_s \cong U(L_s)$ for some Lie algebra L_s acting as derivations of R_s, and $p_s \in H_s^*$ defined by $p_s(h) = \epsilon_s(f_s \cdot h)$ for $h \in H_s$ is differentially produced by R_s. (See [1] for a realization theorem for such control systems.) The triple (H_s, R_s, f_s) represents a continuous control system.

We describe how the data given above can be used to construct a hybrid control system. Let $H_0 = \coprod_{s \in S} H_s$, the free product of the H_s. Then H_0 is a pointed irreducible bialgebra, since it is generated as an algebra by the primitive elements of the H_s. Let $R = \bigoplus_{s \in S} R_s$. Then R is a commutative k-algebra and a right H-module algebra as follows: the maps $H_0 \to H_s$ induced

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by the maps $\phi_{s',s} : H_{s'} \to H_s$ given by

$$\phi_{s',s} = \begin{cases} \mathrm{id} & \text{if } s' = s \\ \eta_s \circ \epsilon_{s'} & \text{if } s' \neq s \end{cases}$$

induces a bialgebra homomorphism $H_0 \to H_s$. Pullback along this map makes R_s into a right H_0-module algebra. In concrete terms, H_0 is the algebra freely generated by the elements of the Lie algebras L_s, subject to the relations which hold in L_s, and the elements of $L_{s'}$, for $s' \neq s$, act trivially on R_s. We make R into a right H_0-module algebra by allowing H_0 to act component-wise on $R = \bigoplus_{s \in S} R_s$. Denote the action of H_0 on R by $r \cdot h = rA(h)$. Define the augmentation on R by $\alpha = \alpha_{s_0} \oplus 0 \oplus \cdots \oplus 0$, where α_{s_0} is the augmentation on R_{s_0}, and define the observation $f = \sum_{s \in S} f_s$.

Suppose that for each $a \in \Sigma$ and $s \in S$ we are given a bialgebra homomorphism $T_{a,s} : H_s \to H_{a \cdot s}$. Since W is freely generated by Σ, this gives a bialgebra homomorphism $T_{w,s} : H_s \to H_{w \cdot s}$ for each $s \in S$ and $w \in W$. The homomorphisms $H_s \to H_{w \cdot s} \to H_0$ induce a bialgebra endomorphism $H_0 \to H_0$. This gives a semigroup homomorphism $T : W \to \mathrm{End}_{\mathrm{bialg}}\, H_0$. Define the bialgebra

$$H = H_0 \sharp_T kW.$$

See [2] or [3] for a detailed definition of the semidirect product $\sharp$ of a bialgebra by a semigroup algebra; the multiplication is given by

$$(h \sharp w)(h' \sharp w') = h(h'T(w)) \sharp ww'.$$

Suppose that for each $a \in \Sigma$ we are given an algebra homomorphism $R \to R$ mapping $r \mapsto rQ(a)$ such that $R_sQ(a) \subseteq \sum_{a \cdot t = s} R_t$, and $\alpha_t \circ p_t(1_sQ(a)) = 1$, where 1_s is the identity of R_s, $p_t : R \to R_t$ is the projection of R onto R_t, and α_t is the augmentation of R_t, whenever $a \cdot s = t$. (These conditions say that the action of Σ on R reflects the action of Σ on the automaton S.) Since W is freely generated by Σ, this gives a semigroup homomorphism $Q : W \to \mathrm{End}_{\mathrm{alg}}\, R$ such that

$$R_sQ(w) \subseteq \sum_{w \cdot t = s} R_t \tag{1}$$

for all $s \in S$ and $w \in W$, and

$$\alpha_t \circ p_t(1_sQ(w)) = 1, \tag{2}$$

whenever $w \cdot s = t$.

Assume that A, Q, and T satisfy the following compatibility condition.

$$Q(w)A(h) = A(T(w)h)Q(w) \tag{3}$$

for all $h \in H_0$ and $w \in W$. Then defining

$$r \cdot (h \sharp w) = rA(h)Q(w)$$

for all $r \in R$, $h \in H_0$, and $w \in W$ gives R a right H-module structure.

To see that R is a right H-module, we compute

$$\begin{aligned}
(r \cdot (h \sharp w)) \cdot (h' \sharp w') &= rA(h)Q(w) \cdot (h' \sharp w') \\
&= rA(h)Q(w)A(h')Q(w') \\
&= rA(h)A(T(w)h')Q(w)Q(w') \\
&= rA(hT(w)h')Q(ww') \\
&= r \cdot (hT(w)h' \sharp ww') \\
&= r \cdot ((h \sharp w)(h' \sharp w')),
\end{aligned}$$

for all $r \in R$, h, $h' \in H_0$, and w, $w' \in W$. That R is an H-module algebra follows from the facts that R is an H_0-module algebra and that Q maps W to the semigroup of algebra endomorphisms of R. The element $p \in H^*$ defined by $p(h) = \epsilon(f \cdot h)$ for $h \in H$ is the generating series associated with the dynamical system (H, R, f).

Some examples

In this section we give some examples showing how traditional dynamical systems fit into our scheme, and give an example of a simple hybrid system.

Example — continuous systems

Let $\Sigma = \emptyset$ and $S = \{s_0\}$. We then get a continuous dynamical system as described in [1].

Example — discrete systems

Let S be a finite automaton over the alphabet Σ, let $\delta : \Sigma \times S \rightarrow S$ its transition function, $s_0 \in S$ its initial state, and let $F \subseteq S$ a set of accepting states. Let $R_s = H_s = k$ for all $s \in S$, and let

$$f_s = \begin{cases} 1 & \text{if } s \in F \\ 0 & \text{if } s \notin F. \end{cases}$$

Then $H_0 = k$, the homomorphism $T(w) : H_0 \rightarrow H_0$ is the identity for all $w \in W$, $R = k^S$ the algebra of functions from S to k, and $A(h)$ is scalar multiplication of $h \in H_0 = k$ on R. The semigroup W acts on the set S, so it acts on R via the transpose of the action on S: if $r \in R$, then $rQ(w)(s) = r(w \cdot s)$. Viewing R as the algebra of functions from S to k, the observation $f \in R$ is simply the characteristic function of the set of accepting states. It is easily checked that conditions (1), (2), and (3) are satisfied.

If $L \subseteq W$ is the language accepted by the automaton S, then $w \in L$ if and only if $w \cdot s_0 \in F$ if and only if $f(w \cdot s_0) = 1$ if and only if $p(w) = \epsilon(f \cdot w) = 1$. Therefore the generating series p in this case is the characteristic function of the language accepted by the automaton S.

Example — a simple hybrid system

Let $S = \{s_1, s_2\}$ with initial state s_1, $\Sigma = \{a_1, a_2\}$ with action of Σ on S given by $a_i \cdot s_j = s_i$, and let $R_{s_i} = k[X_1, \ldots, X_N]$, and $H_{s_i} = k\langle E_{i1}, E_{i2}\rangle$, where E_{ij} act as derivations on R_{s_i}. For simplicity we write R_i for R_{s_i} and H_i for H_{s_i}. Note that $H_0 = k\langle E_{11}, E_{12}, E_{21}, E_{22}\rangle$, and that $R = k[X_1, \ldots, X_N] \oplus k[X_1, \ldots, X_N]$. Let $\alpha_i : R_i \rightarrow k$ be the augmentation mapping $p \mapsto p(0)$. Denote $\hat{1} = 2$ and $\hat{2} = 1$.

The map $T(a_i) : H_0 \rightarrow H_0$ is induced by the homomorphism

$$\begin{aligned} E_{ij} &\mapsto E_{ij} \\ E_{\hat{i}j} &\mapsto 0 \end{aligned}$$

The map $Q(a_i) : R \rightarrow R$ is defined as follows. Recall that $R_i \cong R_j$ (in fact they are equal); let $\rho_{ij} : R_i \rightarrow R_j$ be this isomorphism. Then

$$Q(a_i)(r) = \begin{cases} \rho_{ii}(r) \oplus \alpha_{\hat{i}}(\rho_{i\hat{i}}(r))1_{\hat{i}} & \text{if } r \in R_i \\ 0 & \text{if } r \in R_{\hat{i}}. \end{cases}$$

It is easily checked that conditions (1), (2), and (3) are satisfied.

The correspondence between the Heisenberg representation and the state space representation

In this section we describe the correspondence between the representation of a dynamical system using a bialgebra H and an H-module algebra R (the Heisenberg representation), and the representation of a dynamical system using (in the continuous case) the state space $V \cong \mathbf{R}^N$ and a differential operator on the algebra of polynomial functions R on V, or (in the discrete case) the finite state space S and a semigroup W of words acting on S (the state space representation).

References

[1] R. Grossman and R. G. Larson, The realization of input–output maps using bialgebras, *Forum Math.*, to appear.

[2] R. G. Larson, Cocommutative Hopf algebras, *Canad. J. Math.* **19** (1967), 350–360.

[3] M. E. Sweedler, *Hopf algebras,* W. A. Benjamin, New York, 1969.